

THE VALIDITY OF THE ANALOG OF THE RIEMANN HYPOTHESIS FOR SOME PARTS OF $\zeta(s)$ AND THE NEW FORMULA FOR $\pi(x)$

JAN MOSER

ABSTRACT. An analog of the Riemann hypothesis is proved in this paper. Some new integral equations for the functions $\pi(x)$ and $R(x)$ follows. A new effect that is shown is that these function - with essentially different behavior - are the solutions of the similar integral equations.

This paper is the English version of the paper of reference [1].

1. THE MAIN RESULT

1.1. Let (comp. [2], (7), (21); $2P\beta < \ln P_0$)

$$(1.1) \quad P = (\ln P_0)^{1-\epsilon}, \quad \beta = \left\lfloor \ln^{\frac{2\epsilon}{3}} P_0 \right\rfloor, \quad P_0 = \sqrt{\frac{T}{2\pi}},$$

$0 < \epsilon$ is arbitrarily small and (p is the prime)

$$(1.2) \quad \begin{aligned} \zeta_1(s) &= \prod_{p \leq P} \sum_{k=0}^{\beta} \frac{1}{p^{sk}} = \sum_{n < P_0, p \leq P} \frac{1}{n^s} = \sum'_{n < P_0} \frac{1}{n^s}, \\ \zeta_2(s) &= \prod_{p \leq P} \sum_{k=0}^{\beta} \frac{1}{p^{(1-s)k}} = \sum_{n < P_0, p \leq P} \frac{1}{n^{1-s}} = \sum'_{n < P_0} \frac{1}{n^{1-s}}, \\ \zeta_3(s) &= \chi(s) \zeta_2(s) \end{aligned}$$

where (see [3], p. 16)

$$(1.3) \quad \chi(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}, \quad s \neq 2k+1, \quad k = 0, 1, 2, \dots,$$

and $s = \sigma + it \in \mathbb{C}$. We define the function $\tilde{\zeta}(s)$ as follows

$$(1.4) \quad \begin{aligned} \tilde{\zeta}(s) &= \tilde{\zeta}(s; P, \beta) = \zeta_1(s) + \zeta_3(s) = \\ &= \sum'_{n < P_0} \frac{1}{n^s} + \chi(s) \sum'_{n < P_0} \frac{1}{n^{1-s}}, \quad s \in \mathbb{C}, \quad s \neq 2k+1. \end{aligned}$$

Since

$$\tilde{\zeta}(1-s) = \sum'_{n < P_0} \frac{1}{n^{1-s}} + \chi(1-s) \sum'_{n < P_0} \frac{1}{n^s},$$

Key words and phrases. Riemann zeta-function.

and (see [3], p. 16) $\chi(s)\chi(1-s) = 1$, then

$$\tilde{\zeta}(s) = \chi(s)\tilde{\zeta}(1-s), \quad s \in \mathbb{C}, \quad s \neq 2k+1.$$

Remark 1. The function $\tilde{\zeta}(s)$ obeys the functional equation

$$\tilde{\zeta}(s) = \chi(s)\tilde{\zeta}(1-s)$$

hence, the zeros of $\tilde{\zeta}(s)$ either lie on the critical line $\sigma = \frac{1}{2}$ or occur in pairs symmetrical about this line.

1.2. Since (comp. [3], p. 79)

$$\chi\left(\frac{1}{2} + it\right) = e^{-i2\vartheta(t)}$$

then from (1.4) the formula

$$\begin{aligned} (1.5) \quad e^{i\vartheta(t)}\tilde{\zeta}\left(\frac{1}{2} + it\right) &= \sum'_{n < P_0} \frac{e^{i\{\vartheta(t) - t \ln n\}}}{\sqrt{n}} + \sum'_{n < P_0} \frac{e^{-i\{\vartheta(t) - t \ln n\}}}{\sqrt{n}} = \\ &= 2 \sum'_{n < P_0} \frac{1}{\sqrt{n}} \cos\{\vartheta(t) - t \ln n\} = Z_1(t; P, \beta) \end{aligned}$$

follows. We have studied the zeros of $Z_1(t)$, i.e. the zeros of $\tilde{\zeta}(s)$, on the critical line in the paper [2]. Let

$$\begin{aligned} (1.6) \quad D = D(T, H, K) &= \{s : \sigma \in [-K, K], \quad t \in [T, T+H]\}, \\ K > 1, \quad T > 0, \quad H &\leq \sqrt{T}. \end{aligned}$$

In this paper we prove the following theorem.

Theorem.

$$(1.7) \quad \tilde{\zeta}(s) \neq 0, \quad s \in D, \quad \sigma \neq \frac{1}{2}$$

for all sufficiently big $T > 0$, i.e. for $\tilde{\zeta}(s)$, $s \in D$, $T \rightarrow \infty$ the analog of the Riemann hypothesis is true.

Let us remind the approximate functional equation of Riemann-Hardy-Littlewood ([3], p. 69)

$$(1.8) \quad \zeta(s) = \sum_{n \leq t'} \frac{1}{n^s} + \chi(s) \sum_{n \leq t'} \frac{1}{n^{1-s}} + \mathcal{O}(t^{-\frac{\sigma}{2}}), \quad t' = \sqrt{\frac{t}{2\pi}},$$

and the Riemann-Siegel formula (comp. (1.5))

$$\begin{aligned} (1.9) \quad e^{i\vartheta(t)}\zeta\left(\frac{1}{2} + it\right) &= Z(t) = 2 \sum_{n \leq t'} \frac{1}{\sqrt{n}} \cos\{\vartheta(t) - t \ln n\} + \mathcal{O}(t^{-\frac{1}{4}}) = \\ &= 2 \sum_{n < P_0} \frac{1}{\sqrt{n}} \cos\{\vartheta(t) - t \ln n\} + \mathcal{O}(T^{-\frac{1}{4}}) + \mathcal{O}(HT^{-\frac{3}{4}}), \quad t \in [T, T+H]. \end{aligned}$$

Remark 2. The term *the part of the function* $\zeta(s)$ is specified by the comparison of the formulae (1.4), (1.8). Next, the condition $H \leq \sqrt{T}$ is related with (1.9).

2. THE FORMULAE FOR SOME PARTS OF $\tilde{\zeta}(s)$

We have (see (1.2))

$$\zeta_1(s) = B_1(s)e^{i\psi_1(s)}, \quad B_1(s) = |\zeta_1(s)| > 0, \quad \sigma > 0,$$

where

$$(2.1) \quad \begin{aligned} B_1(s) &= \prod_{p \leq P} |M_1(p; s, \beta)|, \quad \psi_1(s) = \sum_{p \leq P} \arg\{M_1(p; s, \beta)\}, \\ M_1(p) &= \frac{1 - Q_1^{\beta+1}}{1 - Q_1}, \quad Q_1 = Q_1(p; s) = \frac{1}{p^s}, \quad |Q_1| = \frac{1}{p^\sigma} < 1, \end{aligned}$$

and similarly,

$$\zeta_2(s) = B_2(s)e^{i\psi_2(s)}, \quad B_2(s) > 0, \quad \sigma < 1,$$

where

$$(2.2) \quad \begin{aligned} B_2(s) &= \prod_{p \leq P} |M_2(p)|, \quad \psi_2(s) = \sum_{p \leq P} \arg\{M_2(p)\}, \\ M_2(p) &= \frac{1 - Q_2^{\beta+1}}{1 - Q_2}, \quad Q_2 = \frac{1}{p^{1-s}}, \quad |Q_2| = \frac{1}{p^{1-\sigma}} < 1. \end{aligned}$$

Next, we have (see [3], pp. 68, 79, 329)

$$(2.3) \quad \chi(s) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma} e^{-i2\vartheta(t)} \left\{1 + \mathcal{O}\left(\frac{1}{t}\right)\right\},$$

i.e.

$$\chi(s) = |\chi(s)|e^{i\psi_3(s)}$$

where

$$(2.4) \quad \begin{aligned} |\chi(s)| &= \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma} \{1 + \mathcal{O}\}, \quad |\chi(s)| > 0, \quad s \in D, \\ \psi_3(s) &= -2\vartheta(t) + \mathcal{O}\left(\frac{1}{t}\right), \quad T \rightarrow \infty. \end{aligned}$$

Consequently, we obtain the following formulae

$$(2.5) \quad \begin{aligned} \tilde{\zeta}(s) &= B_1(s)e^{i\psi(s)} + B_2(s)|\chi(s)|e^{i\psi_4(s)}, \\ \psi_4(s) &= \psi_2(s) + \psi_3(s), \quad a \in D \cap \{0 < \sigma < 1\}, \quad T \rightarrow \infty. \end{aligned}$$

Remark 3. Let us remind that the formula (2.3) is connected with the Stirling's formula for $\ln \Gamma(z)$, $z \in \mathbb{C}$ to which corresponds arbitrary fixed strip $-K \leq \sigma \leq K$ (comp. [3], p. 68).

3. THE LEMMAS ON $B_1(s)$, $B_2(s)$

3.1. Let

$$(3.1) \quad D_1(\Delta) = \left\{s : \sigma \in \left[\frac{1}{2} + \Delta, 1 - \Delta\right], \quad t \in [T, T + H]\right\}, \quad \Delta \in \left(0, \frac{1}{4}\right).$$

The following lemma holds true.

Lemma 1.

$$(3.2) \quad \exp\left(-\frac{A}{\Delta}p^{\frac{1}{2}-\Delta}\right) < B_1(s) < \exp\left(\frac{A}{\Delta}p^{\frac{1}{2}-\Delta}\right), \quad s \in D_1(\Delta), \quad T \rightarrow \infty.$$

Proof. We have (see (2.1))

$$\begin{aligned} |M_1| &= \left| 1 - \frac{1}{p^{(\beta+1)s}} \right| \left| 1 - \frac{1}{p^s} \right|^{-1} = \\ &= \left\{ 1 + \frac{1}{p^{2(\beta+1)\sigma}} - \frac{2 \cos\{(\beta+1)\varphi\}}{p^{(\beta+1)\sigma}} \right\}^{\frac{1}{2}} \left\{ 1 + \frac{1}{p^{2\sigma}} - \frac{2 \cos \varphi}{p^\sigma} \right\}^{-\frac{1}{2}} = M_{11} M_{12} \end{aligned}$$

where

$$\varphi = t \ln p.$$

Next, we have (see (1.1))

$$\begin{aligned} \ln M_{11} &= \frac{1}{2} \ln \left\{ 1 + \mathcal{O} \left(\frac{1}{p^{\frac{\beta}{2}}} \right) \right\} = \mathcal{O} \left(\frac{1}{p^{\frac{\beta}{2}}} \right), \\ (3.3) \quad M_{11} &= \exp \left\{ \mathcal{O} \left(\frac{1}{p^{\frac{\beta}{2}}} \right) \right\} \end{aligned}$$

uniformly for $\Delta \in (0, \frac{1}{4})$, and since

$$\frac{1}{2} + \Delta \leq \sigma \leq 1 - \Delta, \quad \frac{1}{p^{2\sigma}} \leq \frac{1}{p^{1+2\Delta}} < 1,$$

then

$$\begin{aligned} \ln M_{12} &= -\frac{1}{2} \ln \left(1 - \frac{1}{p^{2\sigma}} \right) - \frac{1}{2} \ln \left(1 - \frac{2p^\sigma}{p^{2\sigma} + 1} \cos \varphi \right) = \\ &= \frac{1}{p^\sigma} \cos \varphi + \mathcal{O} \left(\frac{1}{p^{2\sigma}} \right), \\ M_{12} &= \exp \left\{ \frac{1}{p^\sigma} \cos \varphi + \mathcal{O} \left(\frac{1}{p^{2\sigma}} \right) \right\}. \end{aligned}$$

Hence (see (2.1)), we have

$$(3.4) \quad B_1(s) = \exp \left\{ \sum_{p \leq P} \frac{1}{p^\sigma} \cos \varphi + \mathcal{O} \left(\sum_{p \leq P} \frac{1}{p^{2\sigma}} \right) \right\}, \quad s \in D_1(\Delta)$$

uniformly for $\Delta \in (0, \frac{1}{4})$. Since

$$\Delta \leq 1 - \sigma \leq \frac{1}{2} - \Delta,$$

then

$$(3.5) \quad \left| \sum_{p \leq P} \frac{1}{p^\sigma} \cos \varphi + \mathcal{O} \left(\sum_{p \leq P} \frac{1}{p^{2\sigma}} \right) \right| < A \sum_{p \leq P} \frac{1}{p^\sigma} < \frac{A}{1-\sigma} p^{1-\sigma} < \frac{A}{\Delta} p^{\frac{1}{2}-\Delta},$$

and from this (see (3.4)) we obtain (3.2). \square

3.2. The following lemma holds true

Lemma 2.

$$(3.6) \quad \exp(-AP^{1-\Delta}) < B_2(s) < \exp(AP^{1-\Delta}), \quad s \in D_1(\Delta), \quad T \rightarrow \infty$$

if the condition

$$(3.7) \quad \Delta\beta > \omega(T)$$

is fulfilled, where $\omega(T)$ increases to ∞ for $T \rightarrow \infty$.

Proof. Since by (3.7), (see (2.2)),

$$(1 - \sigma)(\beta + 1) \geq \Delta(\beta + 1) > \omega(T)$$

then putting $1 - \sigma = \bar{\sigma}$, we obtain the formula

$$(3.8) \quad B_2(s) = \exp \left\{ \sum_{p \leq P} \frac{1}{p^{\bar{\sigma}}} \cos \varphi + \mathcal{O} \left(\sum_{p \leq P} \frac{1}{p^{2\bar{\sigma}}} \right) \right\},$$

(similarly to (3.4)). Since (see (3.1), comp. (3.5); $\Delta \leq \bar{\sigma} \leq \frac{1}{2} - \Delta$)

$$\sum_{p \leq P} \frac{1}{p^{\bar{\sigma}}} + \mathcal{O} \left(\sum_{p \leq P} \frac{1}{p^{2\bar{\sigma}}} \right) = \mathcal{O} \left(\sum_{p \leq P} \frac{1}{p^{\bar{\sigma}}} \right) = \mathcal{O} \left(\frac{P^{1-\bar{\sigma}}}{1-\bar{\sigma}} \right) = \mathcal{O}(P^{1-\Delta}),$$

then we obtain (3.6) from (3.8). \square

Remark 4. The estimate (3.6) is valid in somehow wider domain

$$D_1^+(\Delta) = \left\{ s : \sigma \in \left[\frac{1}{2}, 1 - \Delta \right], t \in [T, T + H] \right\}.$$

4. THE FUNCTION $\tilde{\zeta}(s)$ HAS NO ZERO IN THE RECTANGLE $D_1(\Delta_0)$

First off all (see (2.4))

$$(4.1) \quad |\chi(s)| < \frac{A}{P_0^{2\Delta}}, \quad s \in D_1(\Delta).$$

Next (see (2.5), (3.2), (3.6))

$$\begin{aligned} |\tilde{\zeta}(s)| &\geq B_1(s) - |\chi(s)|B_2(s) > \exp \left(-\frac{A}{\Delta} P^{\frac{1}{2}-\Delta} \right) - \frac{A}{P_0^{2\Delta}} \exp(AP^{1-\Delta}) > \\ &> \exp \left(-\frac{A}{\Delta} P^{\frac{1}{2}-\Delta} \right) - \frac{A}{P_0^{2\Delta}} \exp(AP) = \\ (4.2) \quad &= \left\{ 1 - \frac{A}{P_0^{2\Delta}} \exp \left(AP + \frac{A}{\Delta} P^{\frac{1}{2}-\Delta} \right) \right\} \exp \left(-\frac{A}{\Delta} P^{\frac{1}{2}-\Delta} \right) > \\ &> \left\{ 1 - \frac{A}{P_0^{2\Delta}} \exp \left(\frac{2A}{\Delta} P \right) \right\} \exp \left(-\frac{A}{\Delta} P^{\frac{1}{2}-\Delta} \right) = \\ &= \left\{ 1 - A \exp \left(\frac{2A}{\Delta} P - 2\Delta \ln P_0 \right) \right\} \exp \left(-\frac{A}{\Delta} P^{\frac{1}{2}-\Delta} \right). \end{aligned}$$

Since

$$2\Delta \ln P_0 - \frac{2A}{\Delta} P = \frac{2 \ln P_0}{\Delta} \left(\Delta^2 - A \frac{P}{\ln P_0} \right)$$

then we put (see (1.1))

$$(4.3) \quad \Delta_0 = \Delta_0(T, \epsilon) = \left(2A \frac{P}{\ln P_0} \right)^{\frac{1}{2}} = \frac{\sqrt{2A}}{(\ln P_0)^{\frac{1}{2}}}.$$

Because (see (1.1))

$$\Delta_0 \beta > A_1 (\ln P_0)^{\frac{\epsilon}{6}} \rightarrow \infty, \quad T \rightarrow \infty$$

then the condition (3.7) is fulfilled. Hence, we obtain from (4.2) by (1.1) and (4.3) the estimate

$$|\tilde{\zeta}(s)| > \frac{1}{2} \exp\left(-\frac{A}{\Delta_0} P^{\frac{1}{2}-\Delta_0}\right) > \exp\left(-\frac{A}{\Delta_0} P^{\frac{1}{2}}\right) = \exp\left(-\sqrt{\frac{A}{2} \ln P_0}\right),$$

$$s \in D_1(\Delta_0), T \rightarrow \infty.$$

Namely, we have the following lemma holds true.

Lemma 3.

$$|\tilde{\zeta}(s)| > e^{-\sqrt{A \ln P_0}}, \quad s \in D_1(\Delta_0), \quad T \rightarrow \infty.$$

Corollary 1.

$$(4.4) \quad \tilde{\zeta}(s) \neq 0, \quad s \in D_1(\Delta_0), \quad T \rightarrow \infty.$$

5. THE FUNCTION $\tilde{\zeta}(s)$ HAS NO ZERO IN THE RECTANGLE $D_2(\Delta_0)$

Let

$$D_2(\Delta_0) = \{s : \sigma \in [1 - \Delta_0, K], \quad t \in [T, T + K]\}.$$

We remark that the formula (3.4) is valid for all $\sigma \in [1 - \Delta_0, K]$, see the proof of the Lemma 1. Since in our case (comp. (3.5))

$$\left| \sum_{p \leq P} \frac{1}{p^\sigma} \cos \varphi + \mathcal{O}\left(\sum_{p \leq P} \frac{1}{p^{2\sigma}}\right) \right| < A \sum_{p \leq P} \frac{1}{p^{1-\Delta_0}} < \frac{A}{\Delta_0} P^{\Delta_0},$$

then we obtain the estimate (comp. (3.2))

$$(5.1) \quad \exp\left(-\frac{A}{\Delta_0} P^{\Delta_0}\right) < B_1(s) < \exp\left(\frac{A}{\Delta_0} P^{\Delta_0}\right)$$

for $s \in D_2(\Delta_0)$, $T \rightarrow \infty$.

Next, for $\zeta_2(s)$ we use the formula (see (1.2))

$$\zeta_2(s) = \sum'_{n < P_0} \frac{1}{n^{1-s}}.$$

First of all (see (1.1), (4.3) and (1.2) - the product formula for $\zeta_2(s)$)

$$(5.2) \quad \sum_{n < P_0} 1 = (\beta + 1)^{\pi(P)} = \exp\{\pi(P) \ln(\beta + 1)\} <$$

$$< \exp\left(A(\epsilon) \frac{P}{\ln P} \ln \ln P_0\right) = \exp\{A(\epsilon)(\ln P_0)^{1-\epsilon}\} < \exp(\Delta_0 \ln P_0) = P_0^{\Delta_0}$$

where we have used the upper estimate of Chebyshev for $\pi(x)$. Next, (see (1.1), (2.4))

$$|\chi(s)| < \frac{A}{P_0^{2\sigma-1}}, \quad 1 - \Delta_0 \leq \sigma \leq K.$$

Now:

(A) in the rectangle

$$D_{21}(\Delta_0) = D_2(\Delta_0) \cup \{1 - \Delta_0 \leq \sigma \leq 1\}$$

we have (see (1.2), (5.2); $1 - 2\Delta_0 \leq 2\sigma - 1 \leq 1$)

$$\begin{aligned} \zeta_3(s) &= \chi(s)\zeta_2(s) = \mathcal{O}\left(\frac{1}{P_0^{2\sigma-1}} \sum'_{n < P_0} \frac{1}{n^{1-\sigma}}\right) = \mathcal{O}\left(\frac{1}{P_0^{2\sigma-1}} \sum'_{n < P_0} 1\right) = \\ (5.3) \quad &= \mathcal{O}\left(\frac{1}{P_0^{1-2\Delta_0}} P_0^{\Delta_0}\right) = \mathcal{O}\left(\frac{1}{P_0^{1-3\Delta_0}}\right), \end{aligned}$$

(B) in the rectangle

$$D_{22}(\Delta_0) = D_2(\Delta_0) \cup \{1 < \sigma \leq K\}$$

we have

$$\begin{aligned} \zeta_3(s) &= \mathcal{O}\left(\frac{1}{P_0^{2\sigma-1}} \sum_{n < P_0} \frac{1}{n^{1-\sigma}}\right) = \mathcal{O}\left\{\frac{1}{P_0^\sigma} \sum'_{n < P_0} \left(\frac{n}{P_0}\right)^{\sigma-1}\right\} = \\ (5.4) \quad &= \mathcal{O}\left(\frac{1}{P_0^\sigma} \sum'_{n < P_0} 1\right) = \mathcal{O}\left(\frac{1}{P_0^{1-\Delta_0}}\right). \end{aligned}$$

Consequently (see (5.3), (5.4)), we have

$$(5.5) \quad \zeta_3(s) = \mathcal{O}\left(\frac{1}{P_0^{1-3\Delta_0}}\right), \quad s \in D_2(\Delta_0).$$

Since (see (1.1), (4.3))

$$\frac{A}{\Delta_0} P^{\Delta_0} = \frac{A}{\sqrt{2A_1}} (\ln P_0)^{\frac{\epsilon}{2}} (\ln P_0)^{(1-\epsilon)\Delta_0} < (\ln P_0)^{\frac{2\epsilon}{3}}, \quad T \rightarrow \infty,$$

then (see (2.5), (5.1), (5.5)) we obtain in the domain $D_2(\Delta_0)$

$$\begin{aligned} |\tilde{\zeta}(s)| &\geq B_1(s) - |\zeta_3(s)| > \exp\left(-\frac{A}{\Delta_0} P^{\Delta_0}\right) - \frac{A}{P_0^{1-3\Delta_0}} = \\ &= \left\{1 - \exp\left[\frac{A}{\Delta_0} P^{\Delta_0} - (1 - 3\Delta_0) \ln P_0 + \ln A\right]\right\} \exp\left(-\frac{A}{\Delta_0} P^{\Delta_0}\right) > \\ &> \frac{1}{2} \exp\left[-(\ln P_0)^{\frac{2\epsilon}{3}}\right] > \exp\left[-(\ln P_0)^\epsilon\right], \quad T \rightarrow \infty, \end{aligned}$$

i.e. the following lemma holds true.

Lemma 4.

$$|\tilde{\zeta}(s)| > e^{-(\ln P_0)^\epsilon}, \quad s \in D_2(\Delta_0), \quad T \rightarrow \infty.$$

Corollary 2.

$$(5.6) \quad \tilde{\zeta}(s) \neq 0, \quad s \in D_2(\Delta_0), \quad T \rightarrow \infty.$$

6. LEMMA ON THE DIFFERENCE OF LOGARITHMS

Let

$$(6.1) \quad D_3(\Delta_0) = \left\{ s : \sigma \in \left(\frac{1}{2}, \frac{1}{2} + \Delta_0 \right], t \in [T, T + H] \right\}$$

where

$$\sigma = \frac{1}{2} + \delta, \quad \delta \in (0, \Delta_0).$$

The following lemma holds true.

Lemma 5.

$$(6.2) \quad \ln B_1(s) - \ln B_2(s) = \mathcal{O} \left\{ \delta (\ln P_0)^{\frac{1-\epsilon}{2}} \right\}, \quad s \in D_3(\Delta_0), \quad T \rightarrow \infty.$$

Proof. We have (see (2.1), (2.2))

$$(6.3) \quad \ln B_1(s) - \ln B_2(s) = Y_1 + Y_2$$

where ($|z| = |\bar{z}|$)

$$\begin{aligned} Y_1 &= \sum_{p \leq P} \left\{ \ln \left| 1 - \frac{p^{-i(\beta+1)t}}{p^{(\beta+1)(\frac{1}{2}+\delta)}} \right| - \ln \left| 1 - \frac{p^{-i(\beta+1)t}}{p^{(\beta+1)(\frac{1}{2}-\delta)}} \right| \right\}, \\ Y_2 &= \sum_{p \leq P} \left\{ \ln \left| 1 - \frac{p^{it}}{p^{\frac{1}{2}-\delta}} \right| - \ln \left| 1 - \frac{p^{it}}{p^{\frac{1}{2}+\delta}} \right| \right\}. \end{aligned}$$

Let

$$x = \frac{1}{p^\sigma}, \quad x \in \left[\frac{p^{-\delta}}{\sqrt{p}}, \frac{p^\delta}{\sqrt{p}} \right].$$

It is clear that (see (1.1), (4.3))

$$\begin{aligned} \delta \ln p &= \mathcal{O}(\Delta_0 \ln P) = \mathcal{O} \left\{ \frac{\ln \ln P_0}{(\ln P_0)^{\frac{\epsilon}{2}}} \right\} \rightarrow 0, \quad T \rightarrow \infty, \\ \frac{p^\delta - p^{-\delta}}{\sqrt{p}} &= \mathcal{O} \left(\delta \frac{\ln P_0}{\sqrt{p}} \right). \end{aligned}$$

Next, by the mean-value theorem

$$\begin{aligned} & \ln \left| 1 - \frac{p^{-i(\beta+1)t}}{p^{(\beta+1)(\frac{1}{2}+\delta)}} \right| - \ln \left| 1 - \frac{p^{-i(\beta+1)t}}{p^{(\beta+1)(\frac{1}{2}-\delta)}} \right| = \\ &= \frac{p^{-\delta} - p^\delta}{\sqrt{p}} \frac{d}{dx} \left\{ \ln \left| 1 - x^{\beta+1} p^{-i(\beta+1)t} \right| \right\} \Big|_{x=x_1}, \\ & x_1 = \frac{1}{p^c}, \quad c \in \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta \right). \end{aligned}$$

Since ($\varphi = t \ln p$)

$$\ln \left| 1 - x^{\beta+1} p^{-i(\beta+1)t} \right| = \frac{1}{2} \ln \left(1 + x^{2\beta+2} - 2x^{\beta+1} \cos\{(\beta+1)\varphi\} \right),$$

then (see (1.1))

$$\begin{aligned} \frac{d}{dx} \ln \left| 1 - x^{\beta+1} p^{-i(\beta+1)t} \right| &= \\ &= \frac{1}{2} \frac{(2\beta+2)x^{2\beta+1} - 2(\beta+1)x^\beta \cos\{(\beta+1)\varphi\}}{1 + x^{2\beta+2} - 2x^{\beta+1} \cos\{(\beta+1)\varphi\}} = \mathcal{O}\left(\frac{\beta}{p^{\sigma\beta}}\right) = \mathcal{O}\left(\frac{\beta}{p^{\frac{\beta}{2}}}\right) \end{aligned}$$

where $\sigma > \frac{1}{2}$, and consequently

$$(6.4) \quad Y_1 = \mathcal{O}\left(\delta \sum_{p \leq P} \frac{\beta}{p^{\frac{\beta}{3}}} \frac{\ln p}{\sqrt{p}}\right) = \mathcal{O}\left(\delta \frac{\beta}{2^{\frac{\beta}{6}}} \sum_{p \leq P} \frac{1}{p^{\frac{\beta}{6}}}\right) = \mathcal{O}(\delta).$$

Similarly, we obtain in the case Y_2

$$\ln \left| 1 - \frac{p^{it}}{p^{\frac{1}{2}-\delta}} \right| - \ln \left| 1 - \frac{p^{it}}{p^{\frac{1}{2}+\delta}} \right| = \frac{p^{-\delta} - p^\delta}{2\sqrt{p}} \frac{d}{dx} \ln(1 + x^2 - 2x \cos \varphi) \Big|_{x=x_2}$$

where

$$\frac{d}{dx} \ln(1 + x^2 - 2x \cos \varphi) = \frac{2x - 2 \cos \varphi}{1 + x^2 - 2x \cos \varphi} = \mathcal{O}(1),$$

because

$$1 + x^2 - 2x \cos \varphi \geq (1 - x)^2 > \left(1 - 2^{-\frac{1}{2}+\delta}\right)^2 > \left(1 - 2^{-\frac{1}{3}}\right)^2 > 0.$$

Thus, we have (see (1.1))

$$(6.5) \quad Y_2 = \mathcal{O}\left(\delta \sum_{p \leq P} \frac{\ln p}{\sqrt{p}}\right) = \mathcal{O}(\delta \sqrt{P}) = \mathcal{O}\left\{\delta (\ln P_0)^{\frac{1-\epsilon}{2}}\right\},$$

(the Abel's transformation was used, comp. [2], (31), (33)). Now, (6.2) follows from (6.3) by (6.4), (6.5). \square

7. MORE ACCURATE FORMULA FOR $\ln |\chi(s)|$

The following lemma holds true.

Lemma 6.

$$(7.1) \quad \ln |\chi(s)| = -\left(\sigma - \frac{1}{2}\right) \ln \frac{t}{2\pi} + \mathcal{O}\left(\frac{2\sigma-1}{t}\right), \quad s \in D_3(\Delta_0), \quad T \rightarrow \infty.$$

Proof. Since (see (1.3))

$$|\chi(s)| = \pi^{\sigma-\frac{1}{2}} \left| \frac{\Gamma\left(\frac{1-\sigma}{2} - i\frac{t}{2}\right)}{\Gamma\left(\frac{\sigma}{2} + i\frac{t}{2}\right)} \right| = \pi^{\sigma-\frac{1}{2}} G_1(\sigma, t),$$

where

$$G_1(\sigma, t) > 0, \quad s \in D_3(\Delta_0), \quad T \rightarrow \infty,$$

then

$$(7.2) \quad \ln |\chi(s)| = \left(\sigma - \frac{1}{2}\right) \ln \pi + \ln G_1(\sigma, t) = \left(\sigma - \frac{1}{2}\right) \ln \pi + G_2(\sigma, t),$$

and $G_2(\sigma, t)$ is the analytic function of the real variable σ for arbitrary fixed t if $s \in D_3(\Delta_0)$, $T \rightarrow \infty$. Since

$$\left| \chi\left(\frac{1}{2} + it\right) \right| = 1$$

then

$$G_2\left(\frac{1}{2}, t\right) = 0,$$

and

$$G_2(\sigma, t) = \left(\sigma - \frac{1}{2}\right) G_3(\sigma, t).$$

Now, (see (7.2))

$$(7.3) \quad \ln |\chi(s)| = \left(\sigma - \frac{1}{2}\right) \{\ln \pi + G_3(\sigma, t)\}, \quad s \in D_3(\Delta_0), \quad T \rightarrow \infty.$$

In the case (2.4) we have

$$(7.4) \quad \ln |\chi(s)| = -\left(\sigma - \frac{1}{2}\right) \ln \frac{t}{2\pi} + G_4(\sigma, t), \quad G_4(\sigma, t) = \mathcal{O}\left(\frac{1}{t}\right),$$

under the conditions (7.3) where $G_4(\sigma, t)$ is the analytic function of the real variable σ . Since

$$G_4\left(\frac{1}{2}, t\right) = 0$$

(see (7.3), (7.4)) then

$$(7.5) \quad G_4(\sigma, t) = \left(\sigma - \frac{1}{2}\right) G_5(\sigma, t).$$

Next, by (7.5), the orders of the functions

$$G_4(\sigma, t), \quad G_5(\sigma, t), \quad s \in D_3(\Delta_0), \quad T \rightarrow \infty$$

in the variable t are equal, i.e. (see (7.4))

$$(7.6) \quad G_5(\sigma, t) = \mathcal{O}\left(\frac{1}{t}\right).$$

Now, the formula (7.1) follows from (7.4) by (7.5), (7.6). □

Remark 5. The formula (7.1) can be proved directly, of course.

8. PROOF OF THE THEOREM

8.1. Let

$$\ln \Lambda(s) = \ln B_1(s) - \ln B_2(s) - \ln |\chi(s)|, \quad s \in D_3(\Delta_0), \quad T \rightarrow \infty.$$

Since (see (1.1), (7.1), $\sigma = \frac{1}{2} + \delta$)

$$(8.1) \quad \ln |\chi(s)| = -\delta \ln \frac{t}{2\pi} + \mathcal{O}\left(\frac{\delta}{t}\right) = -2\delta \ln P_0 + \mathcal{O}\left(\frac{\delta H}{T}\right) + \mathcal{O}\left(\frac{\delta}{T}\right)$$

then we obtain (see (1.6), (6.2), (8.1))

$$(8.2) \quad \begin{aligned} \ln \Lambda(s) &= 2\delta \ln P_0 + \mathcal{O}\left\{\delta (\ln P_0)^{\frac{1-\epsilon}{2}}\right\} + \mathcal{O}\left(\frac{\delta}{\sqrt{T}}\right) = \\ &= \delta \left[2 \ln P_0 + \mathcal{O}\left\{(\ln P_0)^{\frac{1-\epsilon}{2}}\right\} + \mathcal{O}\left(\frac{1}{\sqrt{T}}\right)\right] > \delta \ln P_0 > 0. \end{aligned}$$

Consequently,

$$\Lambda(s) > 1, \quad s \in D_3(\Delta_0), \quad T \rightarrow \infty,$$

and (see (2.5))

$$(8.3) \quad \begin{aligned} |\tilde{\zeta}(s)| &\geq B_1(s) - |\chi(s)|B_2(s) = |\chi(s)|B_2(s) \left(\frac{B_1(s)}{|\chi(s)|B_2(s)} - 1 \right) = \\ &= |\chi(s)|B_2(s)[\Lambda(s) - 1] > 0, \quad s \in D_3(\Delta_0), \quad T \rightarrow \infty. \end{aligned}$$

(The inequality $B_2(s) > 0$, $s \in D_3(\Delta_0)$, $T \rightarrow \infty$ follows from Remark 4.) Now, by (4.4), (5.6), (8.3) and from Remark 1, we have (1.7).

8.2. As an addition to the Theorem we obtain an lower estimate for $|\tilde{\zeta}(s)|$, $s \in D_3(\Delta_0)$. Namely, we have

Lemma 7.

$$(8.4) \quad |\tilde{\zeta}(s)| > \frac{1}{P_0} \sinh \left(\frac{\delta}{2} \ln P_0 \right), \quad s \in D_3(\Delta_0), \quad T \rightarrow \infty, \quad \delta \in (0, \Delta_0).$$

Proof. Since (see (1.1), (3.6), Remark 4 and (6.1))

$$B_2(s) > \exp(-AP^{1-\Delta_0}) > \exp(-AP) = \exp\{-A(\ln P_0)^{1-\epsilon}\},$$

and (see (8.1), (8.2))

$$|\chi(s)| > P_0^{-(2+\epsilon)\delta}, \quad \Lambda(s) > P_0^\delta$$

then (see (8.3))

$$\begin{aligned} |\tilde{\zeta}(s)| &> \frac{\exp\{-A(\ln P_0)^{1-\epsilon}\}}{P_0^{(2+\epsilon)\delta}} (P_0^\delta - 1) > \\ &> 2 \frac{\exp\{-A(\ln P_0)^{1-\epsilon}\}}{P_0^{(\frac{3}{2}+\epsilon)\Delta_0}} \sinh \left(\frac{\delta}{2} \ln P_0 \right) > \frac{1}{P_0} \sinh \left(\frac{\delta}{2} \ln P_0 \right), \end{aligned}$$

i.e. (8.4). □

9. A NEW PROPERTY OF THE FUNCTIONS $\pi(x)$, $R(x)$

Let

$$\begin{aligned} D(\Delta_0) &= D(\Delta_0, T, H, K) = \\ &= \left\{ s : \sigma \in \left[\frac{1}{2} + \Delta_0, K \right], \quad t \in [T, T+H] \right\}, \quad \Delta_0 = \frac{A}{(\ln P_0)^{\frac{1}{2}}}. \end{aligned}$$

We can prove the following

Formula 1.

$$(9.1) \quad \ln \tilde{\zeta}(s) = s \int_2^P \frac{\pi(x)}{x(x^s - 1)} dx - \pi(P) \ln \left(1 - \frac{1}{P^s} \right) + \mathcal{O}(e^{-A\beta}),$$

where $s \in D(\Delta_0)$, $T \rightarrow \infty$ and $\mathcal{O}(e^{-A\beta})$ is the estimate of

$$\sum_{p \leq P} \ln \left(1 - \frac{1}{p^{s(\beta+1)}} \right) + \ln \left(1 + \frac{\chi(s)\zeta(s)}{\zeta_1(s)} \right) = \Omega_1(s; P, \beta).$$

Since

$$\pi(x) = \int_0^x \frac{dx}{\ln v} + R(x) = U(x) + R(x),$$

then we obtain from (9.1)

Formula 2.

$$\ln \tilde{\zeta}(s) = s \int_2^P \frac{R(x)}{x(x^s - 1)} dx - R(P) \ln \left(1 - \frac{1}{P^s}\right) + \mathcal{O}(e^{-A\beta}),$$

$$s \in D(\Delta_0), \quad T \rightarrow \infty$$

where $\mathcal{O}(e^{-A\beta})$ is the estimate of

$$\Omega_1(s; P, \beta) + \Omega_2(s; P),$$

and

$$\begin{aligned} \Omega_2(s) &= -s \int_2^P \frac{dx}{\ln x} \int_2^x \frac{dv}{v(v^s - 1)} - U(P) \ln \left(1 - \frac{1}{2^s}\right) = \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} \int_2^P \frac{dx}{x^{(n+1)s} \ln x} = \mathcal{O}\left(\frac{\sqrt{\ln T}}{T}\right). \end{aligned}$$

Thus, the following properties of the functions $\pi(x)$, $R(x)$ holds true:

- (A) the function $\pi(x)$, $x \in [2, P]$ is the solution of the integral equation (for every fixed $s \in D(\Delta_0)$)

$$(9.2) \quad \ln \tilde{\zeta}(s) = s \int_2^P \frac{\Phi(x)}{x(x^s - 1)} dx - \Phi(P) \ln \left(1 - \frac{1}{P^s}\right) + \Omega_1(s),$$

- (B) the function $R(x)$, $x \in [2, P]$ is the solution of the perturbed integral equation

$$(9.3) \quad \ln \tilde{\zeta}(s) = s \int_2^P \frac{\Phi(x)}{x(x^s - 1)} dx - \Phi(P) \ln \left(1 - \frac{1}{P^s}\right) + \Omega_1(s) + \Omega_2(s).$$

Remark 6. Hence, we have a new property of the functions $\pi(x)$ and $R(x)$: these functions are to solutions of the integral equations (9.2) and (9.3), respectively and the mentioned integral equations are close each to other. This property of $\pi(x)$ and $R(x)$ is fully missing in the theory of $\pi(x)$, $R(x)$ based on the Riemann zeta-function.

Let us remind that the behaviour of the functions $\pi(x)$, $R(x)$ is essentially different, $\pi(x) \sim \frac{x}{\ln x}$, $x \rightarrow \infty$, and $R(x)$, $x \rightarrow \infty$ infinitely many times alternates its sign (Littlewood, 1914).

I would like to thank Michal Demetrian for helping me with the electronic version of this work.

REFERENCES

- [1] J. Moser, ‘The validity of the analog of Riemann hypothesis for some parts of $\zeta(s)$ and the new formula for $\pi(x)$ ’, Acta. Arith. 53 (1997), 297-310 (in russian).
- [2] J. Moser, ‘The prime number and the wandering on lattice of the zeros of some parts of the Riemann-Siegel formula’, Math. Slovaca, 48 (1998), No. 1., 1-26, (in russian).
- [3] E.C. Titchmarsh, ‘The theory of the Riemann zeta-function’, Clarendon Press, Oxford, 1951.

DEPARTMENT OF MATHEMATICAL ANALYSIS AND NUMERICAL MATHEMATICS, COMENIUS UNIVERSITY, MLYNSKA DOLINA M105, 842 48 BRATISLAVA, SLOVAKIA
E-mail address: jan.moser@fmph.uniba.sk